UNIVERSITY OF SOUTH ALABAMA
DEPARTMENT OF MATHEMATICS AND STATISTICS

Comprehensive Examination
Abstract Algebra
Summer Quarter 1997

Part I: Do four of the following five problems.

1. Show that a subgroup of a cyclic group is cyclic.

2. Show that every subgroup of index two is a normal subgroup.

3. Let p be a prime. Show that every finite p-group has a non-trivial center.

4. Do the following problems.

(i) Find the cycle type and a representative of each conjugacy class in S_5.
(ii) Show that a normal subgroup of a group G is a union of conjugacy classes in G.
(iii) List all cycle types and a representative for the elements in A_5.

5. Let p, q be primes with $q < p$. Let G be a group of order pq.

(i) Show that G has a normal subgroup of order p.
(ii) Show that G is cyclic whenever q does not divide $p - 1$.

Part II: Do four of the following five problems.

6. Show that every Euclidean Domain is a Principal Ideal Domain.

7. Show that every finite integral domain is a field.

8. Do the following problems.

(i) show that every finite field has prime characteristic.
(ii) Show that every finite field has order p^n, where p is a prime.

9. Let R be a commutative ring with identity. Let I be an ideal of R and define
$R(I) = \{r \in R | r^n \in I \text{ for some positive integer } n.\}$. This is called the **radical** of I.

(i) Show that $R(I)$ is an ideal containing I.
(ii) Show that the radical of a prime ideal is the ideal itself.

10. Let R be a commutative ring with 1. Let $R[[x]]$ denote the ring of formal power series with indeterminate x.

(i) Show that $1 - x$ is a unit and find an inverse.
(ii) Show that $\sum_{n=0}^{\infty} a_n x^n$ is a unit in $R[[x]]$ if and only if a_0 is a unit in R.