UNIVERSITY OF SOUTH ALABAMA
DEPARTMENT OF MATHEMATICS AND STATISTICS

Comprhensive Examination
Real Analysis I & II
Summer Quarter 1997

Instructions: *Choose five questions from below.*

1. Let E be a compact metric space and let $F \subset E$ be closed.

   (a) Show E is complete.
   (b) If $f : E \longrightarrow \mathbf{R}$ is continous, show $f(E)$ is compact.

2. Let a and b be positive. Let $f_{a,b}(x) = x^a \sin(\frac{1}{x^b})$ for $x \neq 0$ and $f(0) = 0$.

   (a) For which values of a and b is $f_{a,b}$ continous?
   (b) For which values of a and b is $f_{a,b}$ differentiable?

3. Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be continuous. Let x_0 be arbitrary and for $n \geq 1$ define $x_n = f(x_{n-1})$. Suppose that $\lim_{n\to\infty} x_n = x$. Show that $f(x) = x$.

4. Define *connected* and *path-connected* . Show that an open connected subset E of $\mathbf{R}^{\,2}$ is path-connected. Hint: If x_0 is an arbitrary point, let $A = \{x : \text{ there is a path from } x_0 tox\}$ and show $A = E$.

5. Suppose $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$.

   (a) Show $\lim_{n\to\infty} a_n b_n = ab$.
   (b) Define $\sigma_n = \frac{1}{n}(a_1 + \cdots a_n)$. Show $\lim_{n\to\infty} \sigma_n = a$.

6. Let $f : [a, b] \longrightarrow \mathbf{R}$ be bounded with only finitely many discontinuities. Show from the definition that f is integrable.

7. Suppose that f_n are continuous on $[a, b]$ and that $f_n \to f$ uniformly on $[a, b]$. Show that $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$.

8. Let $A : \mathbf{R}^{\rm n} \longrightarrow \mathbf{R}^{\rm n}$ be linear. Show that $DA = A$.

9. Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be continuous and let $f_n(x) = f(nx)$ for $n = 1, 2, \cdots$. Assume that $\{f_n\}_{n=1}^{\infty}$ is equicontinuous. Show f is constant.